

EIGENFORM PRODUCT IDENTITIES FOR HILBERT MODULAR FORMS

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ABSTRACT. We prove that amongst all real quadratic fields and all spaces of Hilbert modular forms of full level and of weight 2 or greater, the product of two Hecke eigenforms is not a Hecke eigenform except for finitely many real quadratic fields and finitely many weights. We show that for $\mathbb{Q}(\sqrt{5})$ there are exactly two such identities.

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1. INTRODUCTION

Let E_k be the normalized Eisenstein series of weight k on $SL_2(\mathbb{Z})$. There are many classical identities between these Eisenstein series E_k for different weights k , for instance

$$(1.1) \quad E_8 = 120E_4^2$$

$$(1.2) \quad E_{10} = \frac{5040}{11}E_6E_4$$

$$(1.3) \quad \Delta_{16} = 240E_4\Delta,$$

where Δ_{16} (resp. Δ) is the unique, normalized cuspidal Hecke eigenform of weight 16 (resp. 12) on $SL_2(\mathbb{Z})$ (the numerical constants in the above identities are normalization constants).

These identities provide solutions to the equation

$$(1.4) \quad g = f \cdot h$$

in Hecke eigenforms. For elliptic modular forms of full level, Duke [5] and Ghate [9] independently considered this question and proved that there are precisely 16 such identities (all of these identities were classically known). Let us note that by considering q -expansions it is immediate that a product of two or more normalized cuspidal Hecke eigenforms cannot be a Hecke eigenform. So in (1.4) at most one of f, h can be cuspidal. We say such an *eigenform product identity holds trivially*, if the dimension of the corresponding modular form space or the cusp form space for g is equal to one. All of the 16 identities hold trivially. The proofs of [5, 9] use Rankin-Selberg convolution. Later Ghate [10] considered another type of eigenform product identities, where the eigenforms are a.e. Hecke eigenforms of weight 3 or greater and of squarefree level, and proved that all such identities hold trivially. Emmons [6] considered $\Gamma_0(p)$, with $p \geq 5$ a prime, and classified eigenform product identities for eigenforms away from the level (eigenform for T_m with m coprime to p). Recently Johnson [12] considered such identities for $\Gamma_1(N)$ of weight 2 or greater and found a complete list of 61 eigenform identities, some of which hold non-trivially. In his thesis Beyerl [1], for the full modular group, considered the question when the quotient of two Hecke eigenforms is a modular form.

Inspired by Johnson's approach [12], we consider this question for Hilbert modular forms. We show that product of two Hecke eigenforms over a fixed real quadratic field can be

another Hecke eigenform. For instance we show that for $F = \mathbb{Q}(\sqrt{5})$

$$(1.5) \quad E_4 = 60E_2^2,$$

$$(1.6) \quad h_8 = 120E_2 \cdot h_6,$$

where $E_2 = E_2(1, 1)$, $E_4 = E_4(1, 1)$ are Eisenstein series of parallel weight two (resp. four) with trivial characters, h_6 (resp. h_8) is the unique normalized cuspidal Hecke eigenform of parallel weight six (resp. eight) for $GL_2^+(\mathcal{O}_F)$ (see Theorem 7.4).

Hence identities of the type (1.4) exists for Hilbert modular forms. So it is natural to ask if there are only finitely many such identities amongst Hilbert modular forms. In this paper we will only consider Hilbert modular forms for Hilbert modular groups of full levels and answer this affirmatively.

In fact, a much stronger assertion is true for Hilbert modular forms. Explicitly, we prove that *amongst all real quadratic fields F* the equation (1.4) has only finitely many solutions in Hecke eigenforms of full level and weights 2 or greater:

Theorem 1.7. Over all real quadratic number fields F and all Hecke eigenforms for $GL_2^+(\mathcal{O}_F)$ of integral parallel weight 2 or greater, the equation $g = f \cdot h$ in the triple (g, f, h) has only finitely many solutions.

Identities such as (1.4) provide relations between Fourier coefficients. One of the important observations of [12] is that relations between Fourier coefficients at small primes (or at powers of small primes) can be used to provide effective bounds for such identities. We adapt methods of [12] to the Hilbert modular form situation, but there are new features: for instance we exploit the discriminant of the real quadratic field F , which manifests itself through its presence in the functional equation of L -functions, to effectively bound the number of the real quadratic field for which product identities can exist. As in Johnson's treatment of the classical case, all bounds in this paper are effective and can be used to obtain a complete list of eigenform product identities, provided that we have the structure of the spaces of Hilbert modular forms of small weights for small D (discriminant of F). We are content with the concrete case for $\mathbb{Q}(\sqrt{5})$ for the moment, and prove that there are exactly two such identities (Theorem 7.4), using such effective bounds in the proof of the Theorem 1.7. As in the case of elliptic modular forms of full level, these two identities for $\mathbb{Q}(\sqrt{5})$ hold trivially.

We remark that for general levels and general narrow ray class characters, the conductors of the characters will appear in the L-values in question. To treat such general situation, one should consider more Fourier coefficients and obtain more equations in the weights to get around of the conductors and finally obstruct such identities.

In Section 2 and 3, we set up the notations and provide the necessary background on Hilbert modular forms of full levels. In Section 4, we prove some formulas on Fourier coefficients of the product of two Hilbert modular forms and break up Theorem 1.7 into Theorem 4.4 and Theorem 4.5, which will be proved in Section 5 and 6 respectively. In Section 7, we obtain the complete list of such identities for $\mathbb{Q}(\sqrt{5})$. Finally, in Conjecture 8.1, we conjecture that our finiteness result (Theorem 1.7) should also hold Hilbert modular forms of weights greater than or equal to two for all totally real fields of any fixed degree and all levels and all narrow ray class characters.

2. PRELIMINARIES

In this section, we set up the notations and recall some necessary notions and results on real quadratic fields that will be used in later sections.

Let $F = \mathbb{Q}(\sqrt{d})$ be a real quadratic field with $d > 1$ being a squarefree integer. Let $\mathcal{O} = \mathcal{O}_F$ be the ring of integers of F , \mathcal{O}^\times the group of units, \mathfrak{d} the different of F , and D the discriminant of F . Therefore $D = d$ if $d \equiv 1 \pmod{4}$ and $D = 4d$ otherwise. Let \mathfrak{p} denote a prime ideal of \mathcal{O} , and $F_{\mathfrak{p}}$ and $\mathcal{O}_{\mathfrak{p}}$ be the completions of F and \mathcal{O} at \mathfrak{p} . For any fractional ideal \mathfrak{c} , considered as a \mathbb{Z} -lattice, we denote \mathfrak{c}^\vee its dual lattice under the trace form of F/\mathbb{Q} ; \mathfrak{c}^\vee is also a fractional ideal. In particular, $\mathcal{O}^\vee = \mathfrak{d}^{-1}$.

We fix one real embedding of F and for $a \in F$, we denote a' the conjugate of a , which gives the other real embedding. Let $F_{\mathbb{R}} = F \otimes_{\mathbb{Q}} \mathbb{R}$, so $a \mapsto (a, a')$ gives the embedding $F \subset F_{\mathbb{R}}$. An element x in $F_{\mathbb{R}}$, hence in F , is called totally positive if its two components are both positive; denoted by $x \gg 0$. For $A \subset F_{\mathbb{R}}$, we denote the subset of totally positive elements by A^+ . Two fractional ideals $\mathfrak{a}, \mathfrak{b}$ are in the same narrow class if $\mathfrak{a} = (a)\mathfrak{b}$ for some $a \gg 0$ in F^\times . We denote the narrow class number of F by h^+ .

Let \mathbb{A} , \mathbb{A}^\times , \mathbb{A}_f and \mathbb{A}_f^\times be the ring of adeles, the group of ideles, the ring of finite adeles and the group of finite ideles, respectively. We recall various characters. A Hecke character ψ is a continuous character on $\mathbb{A}^\times/F^\times$ and $\psi = \prod_v \psi_v$ decomposes uniquely into local characters. We shall denote the induced character on \mathbb{A}^\times also by ψ . An narrow

ideal class character ψ is a Hecke character that is trivial on the subgroup $F^\times F_\mathbb{R}^+ \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times$. Equivalently, in terms of ideals, this is a character on the narrow ideal class group such that $\psi(a\mathcal{O}) = 1$ for all $a \gg 0$ in F . There exists a unique pair $(r, r') \in \{0, 1\}^2$, such that

$$\psi(a\mathcal{O}) = \text{sgn}(a)^r \text{sgn}(a')^{r'}, \text{ for all } a \in F^\times.$$

Note that in general not all sign vectors are associated to a narrow ideal class character. Since the narrow class group is abelian, we have precisely h^+ narrow ideal class characters.

The Dedekind zeta function for F is defined as

$$\zeta_F(s) = \sum_{\mathfrak{m}} N(\mathfrak{m})^{-s} = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1},$$

where \mathfrak{m} is over all nonzero integral ideals and \mathfrak{p} is over all prime ideals in \mathcal{O} . In general, for any narrow ideal class character ψ , we define the Hecke L-function

$$L(s, \psi) = \sum_{\mathfrak{m}} \psi(\mathfrak{m}) N(\mathfrak{m})^{-s} = \prod_{\mathfrak{p}} (1 - \psi(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1}.$$

In particular, $\zeta_F(s) = L(s, 1)$, where we denote the trivial character by 1. The series and the product for $L(s, \psi)$ are absolutely convergent for $\text{Re}(s) > 1$, can be continued to be a meromorphic function on \mathbb{C} , and satisfies a functional equation. More precisely, assuming that the sign vector for ψ is (r, r') , we have the following functional equation

$$(2.1) \quad L(s, \psi) = W(\psi) (\pi^{-2} D)^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{1-s+r}{2}\right) \Gamma\left(\frac{1-s+r'}{2}\right)}{\Gamma\left(\frac{s+r}{2}\right) \Gamma\left(\frac{s+r'}{2}\right)} L(1-s, \bar{\psi}),$$

where $|W(\psi)| = 1$ (See Corollary 8.6, Chapter VII in [14] for details).

The values of $L(s, \psi)$ at $1-k$ with $k \geq 2$, when $r = r' \equiv k \pmod{2}$, are given by

$$(2.2) \quad L(1-k, \psi) = W(\psi) \frac{2}{\pi} \left(\frac{D}{4\pi^2} \right)^{k-\frac{1}{2}} \Gamma(k)^2 L(k, \bar{\psi}).$$

In particular, $L(1-k, \psi) \neq 0$. Moreover, since for any ψ ,

$$\zeta(4k)/\zeta^2(k) \leq \zeta_F(2k)/\zeta_F(k) \leq |L(k, \psi)| \leq \zeta_F(k) \leq \zeta^2(k), \quad k \geq 2,$$

we have the bounds

$$(2.3) \quad \frac{2}{\pi} \left(\frac{D}{4\pi^2} \right)^{k-\frac{1}{2}} \Gamma(k)^2 \frac{\zeta(4k)}{\zeta^2(k)} \leq |L(1-k, \psi)| \leq \frac{2}{\pi} \left(\frac{D}{4\pi^2} \right)^{k-\frac{1}{2}} \Gamma(k)^2 \zeta^2(k).$$

3. HILBERT MODULAR FORMS

We recall the classical and adelic Hilbert modular forms of full levels. It is well-known that the Eisenstein space vanishes if the weight is non-parallel (see, for example, [7, Corollary in Section 1.4]), so we shall only consider parallel weights, since otherwise no such identities exist. Materials in this section can be found in [7] and [15], and we note that our notion of congruence subgroups are more restrictive.

A (Hilbert) congruence subgroup Γ is a subgroup of $\mathrm{GL}_2(F)$ such that there exists an open compact subgroup $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ with $\Gamma = \mathrm{GL}_2(F) \cap \mathrm{GL}_2^+(F_{\mathbb{R}})K$, where $+$ means the determinant is totally positive. It is clear that Γ and K determines each other. For a fractional ideal \mathfrak{c} and an integral ideal \mathfrak{n} in F , we set

$$\Gamma_0(\mathfrak{c}, \mathfrak{n}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O} & \mathfrak{c}^{-1} \\ \mathfrak{n}\mathfrak{c} & \mathcal{O} \end{pmatrix} : \det(\gamma) \in \mathcal{O}^{\times+} \right\}.$$

Here \mathfrak{n} is called the level. It is easy to see that $\Gamma_0(\mathfrak{c}, \mathfrak{n})$ is a congruence subgroup and we denote the corresponding compact open subgroup by $K_0(\mathfrak{c}, \mathfrak{n})$. Denote

$$\gamma^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and it defines an involution on $\mathrm{Mat}_2(\mathbb{A})$, under which $\Gamma_0(\mathfrak{c}, \mathfrak{n})$ and $K_0(\mathfrak{c}, \mathfrak{n})$ are invariant.

We shall be only interested in the full-level groups $\Gamma_0(\mathfrak{c}, \mathcal{O})$. Denote $\Gamma = \Gamma_0(\mathfrak{c}, \mathcal{O})$ for the moment. Let $\mathbb{H}^2 = \{z = (z_1, z_2) : \mathrm{Im}(z_i) > 0, i = 1, 2\}$, and for any element

$$g = (g_1, g_2) = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in \mathrm{GL}_2^+(F_{\mathbb{R}}),$$

set

$$j(g, z) = (c_1 z_1 + d_1)(c_2 z_2 + d_2), \quad gz = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2} \right).$$

Via the embedding $\Gamma \subset \mathrm{GL}_2^+(F_{\mathbb{R}})$ by $\gamma \mapsto (\gamma, \gamma')$, we have an action of Γ on \mathbb{H}^2 ; here γ' is obtained by taking conjugates of all entries of γ . A Hilbert modular form for Γ of parallel weight $k \in \mathbb{Z}$, is a holomorphic function f on \mathbb{H}^2 such that $f|_k \gamma(z) = f(z)$ for any $\gamma \in \Gamma$ and $z \in \mathbb{H}^2$; here the slash- k operator (denoted $|_k$) is defined as

$$f|_k \gamma(z) = (\det(\gamma \gamma'))^{\frac{k}{2}} j(\gamma, z)^{-k} f(\gamma z), \quad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We denote the space of such forms by $M_k(\Gamma)$. Any $f \in M_k(\Gamma)$ admits a Fourier expansion of the form

$$f(z) = \sum_{\mu \in (\mathfrak{c}^{-1})^\vee} a(\mu) \exp(2\pi i \operatorname{Tr}(\mu z)) = \sum_{\mu \in (\mathfrak{c}^{-1})^\vee} a(\mu) q^\mu,$$

where $\operatorname{Tr}(\mu z) = \mu z_1 + \mu' z_2$, $q = (q_1, q_2) = (e^{2\pi i z_1}, e^{2\pi i z_2})$, $q^\mu = q_1^\mu q_2^{\mu'}$. The *Koecher principle* says that $a(\mu) \neq 0 \Rightarrow \mu = 0$ or $\mu \gg 0$. Moreover for any $\varepsilon \in \mathcal{O}^{\times+}$ and any $\mu \in (\mathfrak{c}^{-1})^\vee$, we have $a(\varepsilon \mu) = N(\varepsilon)^{\frac{k}{2}} a(\mu)$. Similar results hold for all congruence subgroups. We call $f \in M_k(\Gamma)$ cuspidal if $a_\gamma(0) = 0$ for any $\gamma \in \operatorname{GL}_2^+(F)$ with $a_\gamma(\mu)$ the Fourier coefficient of $f|_k \gamma$ (which is a Hilbert modular form for the congruence subgroup $\gamma^{-1}\Gamma\gamma$). The space of cusp forms is denoted by $S_k(\Gamma)$. The Petersson inner product is defined by

$$\langle f, h \rangle_\Gamma = \frac{1}{\nu(\Gamma \backslash \mathbb{H}^2)} \int_{\Gamma \backslash \mathbb{H}^2} f(z) \overline{h(z)} (y_1 y_2)^k d\nu(z), \quad d\nu(z) = \prod_{j=1,2} \frac{dx_j dy_j}{y_j^2}, \quad z_j = x_j + i y_j, \quad j = 1, 2.$$

With this, the Eisenstein space $E_k(\Gamma)$ is defined as the orthogonal complement of $S_k(\Gamma)$ in $M_k(\Gamma)$. As in the elliptic case, the Petersson inner product is well-defined if one of the two components is cuspidal.

In general, Hecke theory is not available for $M_k(\Gamma)$ unless $h^+ = 1$. In order to explain the Hecke theory, we need adelic Hilbert modular forms. Now we fix $\Gamma = \Gamma_0(\mathcal{O}, \mathcal{O})$ and $K = K_0(\mathfrak{d}, \mathcal{O})$. Note that K is not the compact open subgroup for Γ and the shift by \mathfrak{d} is for the correct definition of the normalized Fourier coefficients (see below). Set $K_\infty^+ = (\mathbb{R}^\times \operatorname{SO}_2(\mathbb{R}))^2$ and denote also by i the element (i, i) by abuse of notation. An adelic Hilbert modular form of weight k for Γ is a function $f : \operatorname{GL}_2(\mathbb{A}) \rightarrow \mathbb{C}$ such that the following properties hold:

- (1) $f(\gamma g u) = f(g)$ for all $\gamma \in \operatorname{GL}_2(F)$, $g \in \operatorname{GL}_2(\mathbb{A})$, and $u \in K$.
- (2) $f(g u_\infty) = (\det u_\infty)^{\frac{k}{2}} j(u_\infty, i)^{-k} f(g)$ for all $u_\infty \in K_\infty^+$ and $g \in \operatorname{GL}_2(\mathbb{A})$.
- (3) For any $x \in \operatorname{GL}_2(\mathbb{A}_f)$, we define a function $f_x : \mathbb{H}^n \rightarrow \mathbb{C}$ by

$$f_x(z) = (\det g)^{-\frac{k}{2}} j(g, i)^k f(xg)$$

for $g_\infty \in \operatorname{GL}_2^+(\mathbb{R})^2$ such that $g_\infty(i) = z$. Then f_x is a holomorphic function.

- (4) Let U be the unipotent radical of $\operatorname{Res}_{F/\mathbb{Q}} \operatorname{GL}_2$. An adelic Hilbert modular form f is called a cusp form if

$$\int_{U(\mathbb{Q}) \backslash U(\mathbb{A}_\mathbb{Q})} f(ug) du = 0,$$

for all $g \in \mathrm{GL}_2(\mathbb{A})$, where du is a Haar measure on $U(\mathbb{A}_{\mathbb{Q}})$.

We denote the space of holomorphic and cuspidal adelic Hilbert modular forms by \mathcal{M}_k and \mathcal{S}_k respectively. Let ψ be a narrow ideal class character and we say that $f \in \mathcal{M}_k$ has central character ψ if $f(ag) = \psi(a)f(g)$ for each $a \in \mathbb{A}^\times$. The subspace with central character ψ is denoted by $\mathcal{M}_k(\psi)$ and $\mathcal{S}_k(\psi) = \mathcal{S}_k \cap \mathcal{M}_k(\psi)$.

We state the relation between these two versions of Hilbert modular forms. Let

$$\{\mathfrak{c}_\nu := t_\nu \mathcal{O}\}_{\nu=1}^{h^+}$$

be a complete representatives set of the narrow class group of F , with t_ν being finite ideles. We shall assume that $t_1 \mathcal{O}$ represents the identity narrow class. Set $\Gamma_\nu = \Gamma_0(\mathfrak{c}_\nu \mathfrak{d}, \mathcal{O})$. The Petersson inner product is defined by

$$\langle f, h \rangle = \sum_{\nu} \langle f_\nu, h_\nu \rangle_{\Gamma_\nu},$$

with which we define the Eisenstein subspaces \mathcal{E}_k and $\mathcal{E}_k(\psi)$ to be the orthogonal complement of \mathcal{S}_k in \mathcal{M}_k and $\mathcal{S}_k(\psi)$ in $\mathcal{M}_k(\psi)$ respectively.

The following theorem is essentially a special case of Shimura's result [15], where he treated general levels and general narrow ray class characters but did not give the precise definition of the adelic Hilbert modular forms explicitly. The proof is standard, and the argument for elliptic modular forms ([8]) can be carried over without difficulty. See also Dembélé and Cremona's notes [4].

Theorem 3.1 ([15, Shimura]). There exist isomorphisms of complex vector spaces

$$\mathcal{M}_{\mathbf{k}} \simeq \bigoplus_{\nu=1}^{h^+} M_{\mathbf{k}}(\Gamma_\nu), \quad \mathcal{S}_{\mathbf{k}} \simeq \bigoplus_{\nu=1}^{h^+} S_{\mathbf{k}}(\Gamma_\nu) \quad \text{and} \quad \mathcal{E}_{\mathbf{k}} \simeq \bigoplus_{\nu=1}^{h^+} E_{\mathbf{k}}(\Gamma_\nu).$$

Moreover,

$$\mathcal{M}_{\mathbf{k}} = \bigoplus_{\psi} \mathcal{M}_{\mathbf{k}}(\psi), \quad \mathcal{S}_{\mathbf{k}} = \bigoplus_{\psi} \mathcal{S}_{\mathbf{k}}(\psi) \quad \text{and} \quad \mathcal{E}_{\mathbf{k}} = \bigoplus_{\psi} \mathcal{E}_{\mathbf{k}}(\psi),$$

where in all sums ψ runs through all h^+ narrow ideal class characters and some components may vanish.

Under such isomorphisms, we may write an element $f \in \mathcal{M}_{\mathbf{k}}$ as $f = (f_\nu)$ with $f_\nu \in M_{\mathbf{k}}(\Gamma_\nu)$. For each integral ideal \mathfrak{m} , assuming that $\mathfrak{m} = t_\nu^{-1}(\mu)$ with $\mu \in (t_\nu \mathcal{O})^+$, we define

$$c(\mathfrak{m}, f) = N(t_\nu)^{-\frac{k}{2}} a_\nu(\mu),$$

where $a_\nu(\mu)$ is the μ -th normalized Fourier coefficient of f_ν . This is clearly well-defined and we call it the \mathfrak{m} -th Fourier coefficient of f . The normalized constant term $c_\nu(0, f)$, for each ν , is defined to be

$$c_\nu(0, f) = N(t_\nu)^{-\frac{k}{2}} a_\nu(0).$$

It is the space $\mathcal{M}_{\mathbf{k}}$ that carries the Hecke theory. More precisely, for each integral ideal \mathfrak{m} , we have a Hecke operator $T_{\mathfrak{m}}$ on $\mathcal{M}_{\mathbf{k}}$. The Hecke algebra generated by $T_{\mathfrak{m}}$ is commutative and normal and is also generated by $T_{\mathfrak{p}}$ for prime ideals \mathfrak{p} . The subspaces \mathcal{S}_k , \mathcal{E}_k , $\mathcal{S}_k(\psi)$ and $\mathcal{E}_k(\psi)$ are invariant under the Hecke algebra. A Hecke eigenform $f \in \mathcal{M}_{\mathbf{k}}$ is an eigenfunction for all $T_{\mathfrak{m}}$ and we call it normalized if $c(\mathcal{O}, f) = 1$. For a normalized Hecke eigenform, the eigenvalue of $T_{\mathfrak{m}}$ is $c(\mathfrak{m}, f)$ for any \mathfrak{m} . The Hecke multiplicativity properties are similar to those in the case of elliptic modular forms. For example, if $f \in \mathcal{M}_k(\psi)$ is a normalized Hecke eigenform, then $c(\mathfrak{m}\mathfrak{n}, f) = c(\mathfrak{m}, f)c(\mathfrak{n}, f)$ if $(\mathfrak{m}, \mathfrak{n}) = 1$, and if \mathfrak{p} is a prime ideal, then

$$(3.2) \quad c(\mathfrak{p}^2, f) = c(\mathfrak{p}, f)^2 - \psi(\mathfrak{p})N(\mathfrak{p})^{k-1}.$$

The following bound towards the *generalized Ramanujan conjecture*, best so far, was obtained by Kim and Sarnak [13]: if $f \in \mathcal{S}_k$ is a normalized Hecke eigenform and \mathfrak{p} is a prime ideal, then

$$(3.3) \quad |c(\mathfrak{p}, f)| \leq 2N(\mathfrak{p})^{\frac{k-1}{2} + \frac{7}{64}}.$$

This will be needed for the asymptotic behavior of two sides of some equations in the weights, which will obstruct the eigenform identities eventually.

4. PRODUCT OF TWO EIGENFORMS

Assume $k \geq 2$ from now on and keep other notations in the previous sections. We first recall a theorem of Shimura [15] on Eisenstein series. The computation of the constant terms is due to Dasgupta, Darmon and Pollack [3].

Theorem 4.1 ([15, Proposition 3.4], [3, Proposition 2.1]). Let $k \geq 2$ and ϕ and ψ be two narrow ideal class characters and assume that $(\phi_v \psi_v)(-1) = (-1)^k$ for both of the two real places v . There exists an element $E_k(\phi, \psi) \in \mathcal{M}_k(\phi\psi)$ such that

$$c(\mathfrak{m}, E_k(\phi, \psi)) = \sum_{\mathfrak{r}|\mathfrak{m}} \phi(\mathfrak{m}\mathfrak{r}^{-1})\psi(\mathfrak{r})N(\mathfrak{r})^{k-1},$$

for all nonzero integral ideals \mathfrak{m} , and $E_k(\phi, \psi)$ is a normalized eigenform for $T_{\mathfrak{m}}$. Moreover, for each ν ,

$$c_\nu(0, E_k(\phi, \psi)) = 2^{-2}\phi^{-1}(t_\nu)L(\phi^{-1}\psi, 1-k).$$

Corollary 4.2. For any narrow ideal class character ψ , the following set

$$\{E_k(\psi_1, \psi_2) : \psi_1\psi_2 = \psi\}$$

a basis of \mathcal{E}_k consisting of Hecke eigenforms.

Let h denote the class number of F in the following proof and note that h stands for a Hecke eigenform elsewhere.

Proof. When $k = 2$, this is done by Wiles [16, Proposition 1.5]. Assume that $k > 2$. Since the number of cusps is precisely h , by [7, Theorem in Section 1.8], we see that $E_k(\Gamma_\nu)$ has dimension h for each ν , so $\dim(\mathcal{E}_k) = hh^+$, by Theorem 3.1.

On the other hand, there are precisely h narrow class characters ψ with $\psi_\infty(-1) = (-1)^k$, since it is a lift of a fixed character on $F^\times F_{\mathbb{R}} \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times$ to \mathbb{A}^\times , where the index is h . For each such character ψ , by Theorem 4.1, we have h^+ Eisenstein series $E_k(\psi_1, \psi_2)$. Since they are distinct Hecke eigenforms, they are linearly independent. This implies that $\dim(\mathcal{E}_k(\psi)) \geq h^+$, so $\dim(\mathcal{E}_k) \geq hh^+$. This forces that $\dim(\mathcal{E}_k(\psi)) = h^+$ and the corollary follows. \square

We shall need the following elementary lemma on normalized Fourier coefficients of the product of two Hilbert modular forms.

Lemma 4.3. For $j = 1, 2$, let $k_j \in \mathbb{Z}$ and ψ_j be a narrow ideal class character. If $f = (f_\nu) \in \mathcal{M}_{k_1}(\psi_1)$ and $h = (h_\nu) \in \mathcal{M}_{k_2}(\psi_2)$, then $f \cdot h = (f_\nu \cdot h_\nu) \in \mathcal{M}_{k_1+k_2}(\psi_1\psi_2)$. Moreover,

- (1) For each ν , $c_\nu(0, f \cdot h) = c_\nu(0, f)c_\nu(0, h)$.
- (2) $c(\mathcal{O}, f \cdot h) = c(\mathcal{O}, f)c_1(0, h) + c(\mathcal{O}, h)c_1(0, f)$.

$$(3) \quad c((2), f \cdot h) = c_1(0, h)c((2), f) + c(\mathcal{O}, f)c(\mathcal{O}, h) + c_1(0, f)c((2), h).$$

(4) If (2) is inert, then for the ideal (4),

$$\begin{aligned} c((4), f \cdot h) &= c_1(0, h)c((4), f) + c(\mathcal{O}, f)c((3), h) + c((2), f)c((2), h) + c(\mathcal{O}, h)c((3), f) \\ &\quad + c_1(0, f)c((4), h) + \begin{cases} 2c(\mathfrak{d}, f) + 2c(\mathfrak{d}, h) & \text{if } D = 5 \\ 0 & \text{if } D \neq 5 \end{cases}. \end{aligned}$$

(5) For the ideal (3),

$$\begin{aligned} c((3), f \cdot h) &= c_1(0, h)c((3), f) + c(\mathcal{O}, f)c((2), h) + c(\mathcal{O}, h)c((2), f) \\ &\quad + c_1(0, f)c((3), h) + \begin{cases} 2c(\mathfrak{d}, f)c(\mathfrak{d}, h) & \text{if } D = 5 \\ 0 & \text{if } D \neq 5 \end{cases}. \end{aligned}$$

(6) If (2) = \mathfrak{p}^2 (ramifies) or (2) = $\mathfrak{p}\mathfrak{p}'$ (splits), then

$$c(\mathfrak{p}, f \cdot h) = c_\nu(0, h)c(\mathfrak{p}, f) + c_\nu(0, f)c(\mathfrak{p}, h), \quad \mathfrak{p} \sim t_\nu^{-1}\mathcal{O}.$$

Proof. Since it is clear that $f_\nu \cdot h_\nu \in M_{k_1+k_2}(\Gamma_\nu)$, under the isomorphism, the tuple $(f_\nu \cdot h_\nu)$ determines a Hilbert modular form in $\mathcal{M}_{k_1+k_2}$. On the other hand, the function $f \cdot h$ is determined by

$$(f \cdot h)(\alpha_\nu g_\infty) = f(\alpha_\nu g_\infty)h(\alpha_\nu g_\infty) = f_\nu|_{k_1}g_\infty \cdot h_\nu|_{k_2}g_\infty = (f_\nu \cdot h_\nu)|_{k_1+k_2}g_\infty,$$

from which it follows that $f \cdot h = (f_\nu \cdot h_\nu) \in \mathcal{M}_{k_1+k_2}$, hence in $\mathcal{M}_{k_1+k_2}(\psi_1\psi_2)$.

For ease of notations, we assume $t_1 = 1$, so $\mathfrak{c}_1 = \mathcal{O}$. The formula for the constant Fourier coefficients follows directly from the definition and that for the \mathcal{O} -th terms follows from the fact that 1 is minimal in the set \mathcal{O}^+ (of totally positive integers) under the partial order \gg . Indeed, for the component $f_1 \cdot h_1$, the congruence subgroup is $\Gamma_0(\mathfrak{d}, \mathcal{O})$ and $\mathfrak{d}^\vee = \mathcal{O}$ is the lattice where the Fourier expansion sums. Moreover, if $1 = \mu_1 + \mu_2$ with $\mu_1, \mu_2 \in \mathcal{O}^+$, then

$$1 = (\mu_1 + \mu_2)(\mu'_1 + \mu'_2) > \mu_1\mu'_1 + \mu_2\mu'_2 \geq 1 + 1 = 2;$$

a contradiction and the formula follows. For the ideal (2), $\nu = 1$. Then the Fourier expansion sums over \mathcal{O} and we show that if $2 = \mu_1 + \mu_2$ inside \mathcal{O}^+ , then we must have $\mu_1 = \mu_2 = 1$. We see that

$$4 = N(2) = N(\mu_1 + \mu_2) \geq N(\mu_1) + N(\mu_2) + 2\sqrt{N(\mu_1)N(\mu_2)} \geq 1 + 1 + 2 = 4,$$

which forces $N(\mu_1) = N(\mu_2) = 1$ and $\mu_1\mu'_2 = \mu'_1\mu_2$. It follows that $\mu_1 = \mu_2 = 1$.

We now consider the ideal (4) when (2) is inert. We first note that $D = d \equiv 1 \pmod{4}$. Assume $4 = \mu_1 + \mu_2$ with $\mu_1, \mu_2 \gg 0$ and

$$\mu_j = a_j + b_j \frac{1 + \sqrt{D}}{2}, \quad a_j, b_j \in \mathbb{Z}, j = 1, 2.$$

Since $4 = \mu_1 + \mu_2$, we have $b_1 = -b_2$ and $a_1 + a_2 = 4$. Moreover, since $\mu_j \gg 0$, we have

$$a_j + \frac{b_j}{2} > \frac{|b_j|}{2} \sqrt{D}, \quad j = 1, 2.$$

If $b_1 = b_2 = 0$, then we have the three possibilities (1, 3), (2, 2) and (3, 1) for the pair (μ_1, μ_2) . Now we may assume that $b_1 = -b_2 > 0$, and the case when $b_1 < 0$ follows by switching μ_1 and μ_2 . If $D \neq 5$, then $D \geq 13$. It follows that $2a_1 > \sqrt{13} - 1 > 2$ and $2a_2 > \sqrt{13} + 1 > 4$, so $a_1 \geq 2$ and $a_2 \geq 3$. But $a_1 + a_2 = 4$ and we have a contradiction. So if $D \neq 5$, we only have the above three possibilities. If $D = 5$, we first note that $b_1 = 1$, since otherwise $a_1 > \sqrt{5} - 1 > 1$ and $a_2 > \sqrt{5} + 1 > 3$. This implies that $a_1 > 0$ and $a_2 > 1$. Therefore, we have only two cases $(a_1, a_2) = (1, 3)$ or $(2, 2)$. So in total we have four more pairs for (μ_1, μ_2) :

$$\left(\frac{5 + \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2} \right), \quad \left(\frac{5 - \sqrt{5}}{2}, \frac{3 + \sqrt{5}}{2} \right), \quad \left(\frac{3 + \sqrt{5}}{2}, \frac{5 - \sqrt{5}}{2} \right), \quad \left(\frac{3 - \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2} \right).$$

This completes the case by noting that

$$\left(\frac{5 + \sqrt{5}}{2} \right) = \left(\frac{5 - \sqrt{5}}{2} \right) = \mathfrak{d}, \quad \frac{3 + \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2} \in \mathcal{O}^\times.$$

The ideal (3) can be taken care of similarly.

Now assume that $(2) = \mathfrak{p}^2$ ramifies or $(2) = \mathfrak{p}\mathfrak{p}'$ splits, and $\mathfrak{p} = t_\nu^{-1}(\mu)$ with $\mu \in (t_\nu \mathcal{O})^+$. Then the Fourier expansion sums over $t_\nu \mathcal{O}$ and we show that μ is minimal among the totally positive elements in $t_\nu \mathcal{O}$. Indeed, assume otherwise and $\mu = \mu_1 + \mu_2$ with μ_1, μ_2 totally positive. Note first that $N(\mu) = N(t_\nu)N(\mathfrak{p}) = 2N(t_\nu)$. But

$$2N(t_\nu) = N(\mu_1 + \mu_2) \geq N(\mu_1) + N(\mu_2) + 2\sqrt{N(\mu_1\mu_2)} \geq 4N(t_\nu),$$

which is impossible. So μ is minimal and the formula for $c(\mathfrak{p}, f \cdot h)$ follows. \square

We now prove Theorem 1.7. We separate the assertion of Theorem 1.7 in two separate assertions. We assume that f and h are normalized Hecke eigenforms with the set of

normalized Fourier coefficients

$$\{c(\mathbf{m}, f), c_\nu(0, f)\} \quad \text{and} \quad \{c(\mathbf{m}, h), c_\nu(0, h)\}$$

respectively. Note that we are in full level case and all Hecke eigenforms are normalizable. Clearly, we can divide it into two cases: $c_1(0, f)c_1(0, h) \neq 0$ or $c_1(0, f)c_1(0, h) = 0$. Therefore, we have to prove the following two theorems, whose proof will be given in the next two sections.

Theorem 4.4. Among the solutions to the equation $g = f \cdot h$ in the Theorem 1.7, there are finitely many solutions with $c_1(0, f)c_1(0, h) \neq 0$.

Theorem 4.5. Among the solutions to the equation $g = f \cdot h$ in the Theorem 1.7, there are finitely many solutions with $c_1(0, f)c_1(0, h) = 0$.

5. PROOF OF THEOREM 4.4

Assume that f and h are normalized Hecke eigenforms with $c_1(0, f)c_1(0, h) \neq 0$ and $g = f \cdot h$ is also a Hecke eigenform. By Theorem 4.1 and Corollary 4.2, they must be Eisenstein series and we may assume that

$$f = E_{k_1}(\phi_1, \psi_1) \quad \text{and} \quad h = E_{k_2}(\phi_2, \psi_2)$$

with ϕ_j and ψ_j being narrow ideal class characters, $j = 1, 2$. Therefore, by Theorem 4.1, we have

$$c_1(0, f) = 2^{-2}L(1 - k_1, \phi_1^{-1}\psi_1) \quad \text{and} \quad c_1(0, h) = 2^{-2}L(1 - k_2, \phi_2^{-1}\psi_2).$$

By Lemma 4.3 we have $c_\nu(0, g) = c_\nu(0, f)c_\nu(0, h)$ and

$$c(\mathcal{O}, g) = c(\mathcal{O}, f)c_1(0, h) + c(\mathcal{O}, h)c_1(0, f) = c_1(0, h) + c_1(0, f).$$

Since g is a Hecke eigenform, up to a nonzero scalar, g is equal to $E_{k_1+k_2}(\phi, \psi)$ for some ϕ and ψ . By comparing the \mathcal{O} -th terms, we have

$$g = (c_1(0, f) + c_1(0, h))E_{k_1+k_2}(\phi, \psi).$$

Then from the ν -th constant terms, we derive that

$$\frac{c_\nu(0, f)c_\nu(0, h)}{c_1(0, f) + c_1(0, h)} = c_\nu(0, E_{k_1+k_2}(\phi, \psi)).$$

It follows, by Theorem 4.1, that for each ν ,

$$\phi_1(t_\nu)\phi_2(t_\nu)\left(\frac{1}{L(1-k_1, \phi_1^{-1}\psi_1)} + \frac{1}{L(1-k_2, \phi_2^{-1}\psi_2)}\right) = \phi(t_\nu)\frac{1}{L(1-k_1-k_2, \phi^{-1}\psi)}.$$

By considering the case $\nu = 1$, we see that

$$(5.1) \quad \phi_1(t_\nu)\phi_2(t_\nu) = \phi(t_\nu), \text{ for each } \nu,$$

$$(5.2) \quad \frac{1}{L(1-k_1, \phi_1^{-1}\psi_1)} + \frac{1}{L(1-k_2, \phi_2^{-1}\psi_2)} = \frac{1}{L(1-k_1-k_2, \phi^{-1}\psi)}.$$

It follows from (5.1) that $\phi_1\phi_2 = \phi$, so $\psi = \psi_1\psi_2$, since $\phi\psi = \phi_1\phi_2\psi_1\psi_2$.

We now treat the case when $k_1 \neq k_2$. We may assume that $k_1 > k_2$. First note that, if k_1 is large, then

$$\left|\frac{L(1-k_1, \phi_1^{-1}\psi_1)}{L(1-k_2, \phi_2^{-1}\psi_2)}\right| \geq \left(\frac{D}{4\pi^2}\right)^{k_1-k_2} \frac{\Gamma(k_1)^2}{\Gamma(k_2)^2} \frac{\zeta(4k_1)}{\zeta(k_1)^2\zeta(k_2)^2} > \left(\frac{1}{4\pi^2}\right)^{k_1-k_2} \frac{\Gamma(k_1)^2}{\Gamma(k_2)^2} \frac{\zeta(4k_1)}{\zeta(k_1)^2\zeta(k_2)^2},$$

which in turn is bigger than 1; indeed, if $k_2 \geq k_1/2$, then

$$\left(\frac{1}{4\pi^2}\right)^{k_1-k_2} \frac{\Gamma(k_1)^2}{\Gamma(k_2)^2} \geq \left(\frac{k_2^2}{4\pi^2}\right)^{k_1-k_2} > 2,$$

while if $k_2 < k_1/2$, then

$$\left(\frac{1}{4\pi^2}\right)^{k_1-k_2} \frac{\Gamma(k_1)^2}{\Gamma(k_2)^2} \geq \left(\frac{1}{4\pi^2}\right)^{k_1-k_2} \frac{(k_1-1)!^2}{(k_2-1)!(k_1-k_2)!} \geq \frac{(k_1-1)!}{(4\pi^2)^{k_1-k_2}} > 2.$$

From this and by (2.3) and (5.2), if k_1 is large, we have for some constant $C > 0$ independent of k_1, k_2 and D ,

$$\begin{aligned} 1 &= \left| (L(1-k_1, \phi_1^{-1}\psi_1) + L(1-k_2, \phi_2^{-1}\psi_2)) \frac{L(1-k_1-k_2, \phi^{-1}\psi)}{L(1-k_1, \phi_1^{-1}\psi_1)L(1-k_2, \phi_2^{-1}\psi_2)} \right| \\ &\geq C \left(\frac{D}{4\pi^2}\right)^{k_2} \frac{\Gamma(k_1+k_2)^2}{\Gamma(k_1)^2} \left| \left(\frac{Dk_2^2}{4\pi^2}\right)^{k_1-k_2} \frac{\zeta(4k_1)}{\zeta^2(k_1)\zeta^2(k_2)} - 1 \right| \geq C \left(\frac{D}{4\pi^2}\right)^{k_2} \frac{\Gamma(k_1+k_2)^2}{\Gamma(k_1)^2}, \end{aligned}$$

while this last expression can be arbitrarily large if k_1 is large since $\Gamma(k_1+k_2) \geq k_1^{k_2}\Gamma(k_2)$. For each fixed pair (k_1, k_2) , this is also large, thus exceeds 1 if D is large. This finishes the case when $k_1 \neq k_2$.

For the rest of the proof of Theorem 4.4, we assume that $k_1 = k_2$, so $k = 2k_1$. Let us consider more normalized Fourier coefficients to complete the proof.

5.1. The case when $(2) = \mathfrak{p}$ is inert. In particular, \mathfrak{p} is trivial in the narrow ideal class, so all of the narrow ideal class characters are trivial at \mathfrak{p} . In this case, by Lemma 4.3, after simplification and setting $k_1 = k_2$, we have

$$\frac{4}{L(1 - k_1, \phi_1^{-1}\psi_1)L(1 - k_1, \phi_2^{-1}\psi_2)} + \frac{1 + 4^{k_1-1}}{L(1 - k_1, \phi_1^{-1}\psi_1)} + \frac{1 + 4^{k_1-1}}{L(1 - k_1, \phi_2^{-1}\psi_2)} = \frac{1 + 4^{2k_1-1}}{L(1 - 2k_1, \phi^{-1}\psi)},$$

which, together with (5.2), implies that

$$(5.3) \quad \frac{4^{2k_1-1} - 4^{k_1-1}}{L(1 - 2k_1, \phi^{-1}\psi)} = \frac{4}{L(1 - k_1, \phi_1^{-1}\psi_1)L(1 - k_1, \phi_2^{-1}\psi_2)}.$$

However, by (2.3), for a constant $C > 0$ that are independent of k_1 and D , we have

$$\left| \frac{4L(1 - 2k_1, \phi^{-1}\psi)}{L(1 - k_1, \phi_1^{-1}\psi_1)L(1 - k_1, \phi_2^{-1}\psi_2)} \right| \geq C\sqrt{D}(k_1 - 1)4^{2k_1} \geq C(k_1 - 1)4^{2k_1},$$

by the following Stirling's bound on the binomial coefficients

$$\binom{2n}{n} \geq n^{-\frac{1}{2}}2^{2n-1}.$$

Therefore, this, together with (5.3), implies that k_1 is bounded. For each such k_1 , above inequalities also implies that D is bounded, which finishes the proof in this case.

5.2. The case when $(2) = \mathfrak{p}^2$ or $(2) = \mathfrak{p}\mathfrak{p}'$. Assume $\mathfrak{p} = t_\nu^{-1}(\mu)$. Again by Lemma 4.3, we have

$$\phi_1(t_\nu) \frac{\phi_1(\mathfrak{p}) + \psi_1(\mathfrak{p})2^{k_1-1}}{L(1 - k_1, \phi_1^{-1}\psi_1)} + \phi_2(t_\nu) \frac{\phi_2(\mathfrak{p}) + \psi_2(\mathfrak{p})2^{k_1-1}}{L(1 - k_1, \phi_2^{-1}\psi_2)} = \phi(t_\nu) \frac{\phi(\mathfrak{p}) + \psi(\mathfrak{p})2^{2k_1-1}}{L(1 - 2k_1, \phi^{-1}\psi)},$$

which, together with (5.2), implies that

$$(5.4) \quad \frac{B}{L(1 - 2k_1, \phi^{-1}\psi)} = \frac{A}{L(1 - k_1, \phi_1^{-1}\psi_1)}, \quad \text{with}$$

$$(5.5) \quad B = \phi(t_\nu)\psi(\mathfrak{p})2^{2k_1-1} - \phi_2(t_\nu)\psi_2(\mathfrak{p})2^{k_1-1}, \quad A = \phi_1(t_\nu)\psi_1(\mathfrak{p})2^{k_1-1} - \phi_2(t_\nu)\psi_2(\mathfrak{p})2^{k_1-1},$$

since $t_\nu\mathfrak{p} = (\mu)$ and $\phi(t_\nu) = \phi(\mathfrak{p})$ and the same holds for any narrow ideal class character.

Lemma 5.6. There exists a constant $C > 0$, such that $|A| \geq CD^{-\frac{1}{2}}$ for all D, k_1, ϕ_j, ψ_j , $j = 1, 2$.

Proof. If $h^+ = 1$ or 2 , then C is an integer and $|C| \geq 1$. If $h^+ > 2$, we see that

$$|A| \geq 2^{k_1-1} \left| 1 - e^{\frac{2\pi i}{h^+}} \right|.$$

If $2 < h^+ \leq 6$, then clearly $|A| \geq 2$. If $h^+ > 6$, we have

$$|A| \geq 2 \cdot \sin\left(\frac{2\pi}{h^+}\right) \geq \frac{2\pi}{h^+}.$$

Recall the well-known trivial bound of class number of real quadratic fields: there exists a constant $C > 0$, such that $h^+ \leq C\sqrt{D}$ for all D . It follows that $|A| \geq 2\pi C^{-1}D^{-\frac{1}{2}}$. Replacing $2\pi C^{-1}$ with C , we finish the proof. \square

We continue the proof. By (2.3), we see that

$$\left| \frac{L(1-2k_1, \phi^{-1}\psi)}{L(1-k_1, \phi_1^{-1}\psi_1)} \right| \geq C' D^{k_2} \frac{\Gamma(2k_1)^2}{\Gamma(k_1)^2} \geq C' D^{k_2} \Gamma(2k_1),$$

with $C' > 0$ being a constant that is independent of k_1, k_2 and D . By Lemma 5.6,

$$|B| \geq C' C D^{k_2-\frac{1}{2}} \Gamma(2k_1) \geq C' C \Gamma(2k_1).$$

But $|B| \leq 2^{2k_1}$, which forces that there are only finitely many k_1 . Now for each fixed k_1 , such inequalities also shows that there can be only finitely many D , proving this case, hence Theorem 4.4.

We remark that in the ramified case, A is an integer and hence $|A| \geq 1$. In the split case, we may apply the identity $c(\mathfrak{p}, g)c(\mathfrak{p}', g) = c((2), g)$. By lengthy but elementary computation, we may see that if k_1 is large, we must have

$$\phi_1(t_\nu)\psi_1(\mathfrak{p}) = -\phi_2(t_\nu)\psi_2(\mathfrak{p}),$$

from which we also derive $|A| \geq 1$ in this case. In other words, we may avoid Lemma 5.6 and the class number bound.

6. PROOF OF THEOREM 4.5

As before, let f, h be normalized Hecke eigenforms and assume $g = f \cdot h$ is also a Hecke eigenform. To prove Theorem 4.5, assume that $c_1(0, f)c_1(0, h) = 0$. We first note that if $c_1(0, f) = c_1(0, h) = 0$, then by Lemma 4.3, we see that $c(\mathcal{O}, g) = 0$ and g is not a Hecke

eigenform. So one of the factors is an Eisenstein series, thus consider only the parallel weight case. So, we may assume that

$$c_1(0, f) \neq 0 \quad \text{and} \quad c_1(0, h) = 0$$

for the rest of this paper. We observe that h necessarily lie in $\mathcal{S}_{k_2}(\psi_2)$ and $f = E_{k_1}(\phi_1, \psi_1)$ for some narrow ideal class characters ϕ_1, ψ_1, ψ_2 by Theorem 4.1.

Since $c(\mathcal{O}, g) = c_1(0, f)$, we see that $c_1(0, f)^{-1}g$ is a normalized Hecke eigenform. Now by Lemma 4.3, we see that

$$(6.1) \quad \frac{c((2), g)}{c_1(0, f)} = \frac{1}{c_1(0, f)} + c((2), h).$$

By Proposition 2.2 in [15], we know that $c_1(0, f)^{-1}c((2), g)$ and $c((2), h)$ are algebraic integers, so is $\frac{1}{c_1(0, f)}$. But since (2.3) holds for any ψ , it gives a uniform bound for $L(1 - k, \psi)^\sigma$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It follows that for some constant $C > 0$,

$$\left| \frac{1}{c_1(0, f)^\sigma} \right| \leq C \left(\frac{4\pi^2}{D} \right)^{k_1 - \frac{1}{2}} \frac{1}{\Gamma(k_1)^2} \leq C \frac{(4\pi^2)^{k_1 - \frac{1}{2}}}{\Gamma(k_1)^2} \rightarrow 0, \quad \text{as } k_1 \rightarrow \infty.$$

In particular, $|\frac{1}{c_1(0, f)^\sigma}| < 1$ for all σ if k_1 is large, in which case $\frac{1}{c_1(0, f)}$ is not an algebraic integer. The same holds for large D with k_1 being fixed. This proves that for $g = f \cdot h$ to be a Hecke eigenform, there are only finitely many possibilities for D and k_1 , so there are only finitely many possible ϕ_1, ψ_1 and f .

To finish the proof of Theorem 4.5, it suffices to show that for fixed f , there are only finitely many h such that $g = f \cdot h$ is an eigenform. So, with f fixed, we only have to show that k_2 is bounded. We will prove this in the following subsections.

6.1. The case when $(2) = \mathfrak{p}^2$ ramifies. Suppose $\mathfrak{p} \sim t_\nu^{-1}\mathcal{O}$. Then by Lemma 4.3, we have

$$\frac{c(\mathfrak{p}, g)}{c_1(0, f)} = \phi_1(t_\nu)c(\mathfrak{p}, h).$$

This, together with (3.2) and (6.1), implies that

$$\psi_2(\mathfrak{p})2^{k_2-1}(1 - (\phi_1\psi_1)(\mathfrak{p})2^{k_1}) = \frac{1}{c_1(0, f)}.$$

Clearly, this is impossible if k_2 is large.

6.2. The case when $(2) = \mathfrak{p}\mathfrak{p}'$ splits. Suppose $\mathfrak{p} \sim t_\nu^{-1}\mathcal{O}$. Since $c(\mathfrak{p}, h)c(\mathfrak{p}', h) = c((2), h)$ and the same identity holds for $c_1(0, f)^{-1}g$, we have, by Lemma 4.3 and (6.1),

$$\phi_1(t_\nu)c(\mathfrak{p}, h)\phi_1^{-1}(t_\nu)c(\mathfrak{p}', h) = \frac{1}{c_1(0, f)} + c(\mathfrak{p}, h)c(\mathfrak{p}', h),$$

and $\frac{1}{c_1(0, f)} = 0$, which is impossible.

6.3. The final case when (2) is inert. In this case, we need the (4) -th Fourier coefficients. We first assume that $D \neq 5$. By Lemma 4.3, we have

$$\frac{c((2), g)}{c_1(0, f)} = c((2), h) + \frac{1}{c_1(0, f)},$$

$$\frac{c((4), g)}{c_1(0, f)} = c((4), h) + A, \text{ with } A = \frac{c((3), h) + c((3), f) + c((2), h)c((2), f)}{c_1(0, f)}.$$

Moreover,

$$\frac{c((4), g)}{c_1(0, f)} = \left(\frac{c((2), g)}{c_1(0, f)} \right)^2 - 4^{k_1+k_2-1}$$

and $c((4), h) = c((2), h)^2 - 4^{k_2-1}$. It follows that

$$4^{k_2-1}(1 - 4^{k_1}) = -\frac{1}{c_1(0, f)^2} - \frac{2c((2), h)}{c_1(0, f)} + A,$$

which is impossible when k_2 is large, since the right-hand side is bounded by $9^{\frac{k_2}{2}}$ up to a constant.

Finally we treat the case $D = 5$. By Lemma 4.3, we have $c_1(0, f)^{-1}c((4), g) = c((4), h) + B$, with

$$B = \frac{c((3), h) + c((3), f) + c((2), f)c((2), h) + 2c(\mathfrak{d}, h) + 2c(\mathfrak{d}, f)}{c_1(0, f)}.$$

One sees that B is bounded by $9^{\frac{k_2}{2}}$ up to a constant, since $N(\mathfrak{d}) = 5$. By the same argument as above, we have

$$(6.2) \quad 4^{k_2-1}(1 - 4^{k_1}) = -\frac{1}{c_1(0, f)^2} - \frac{2c((2), h)}{c_1(0, f)} + B,$$

which is again not possible if k_2 is large.

This completes the proof of Theorem 4.5, hence that of Theorem 1.7.

7. EIGENFORM PRODUCT IDENTITIES FOR $\mathbb{Q}(\sqrt{5})$

In this section, we consider the concrete case $D = 5$ and find the complete list of eigenform product identities.

The class number is 1 for the field $\mathbb{Q}(\sqrt{5})$, and (2) and (3) both are inert. Since the fundamental unit is $\epsilon_0 = \frac{1+\sqrt{5}}{2}$ which has norm -1 , we have $h^+ = 1$ and $\psi = 1$. We shall drop the characters and denote $E_k = E_k(1, 1)$. The inequality (2.3) implies

$$(7.1) \quad \frac{72}{\pi^5} \left(\frac{5}{4\pi^2} \right)^{k-\frac{1}{2}} \Gamma(k)^2 \leq |\zeta_F(1-k)| \leq \frac{\pi^3}{18} \left(\frac{5}{4\pi^2} \right)^{k-\frac{1}{2}} \Gamma(k)^2,$$

since $1 < \zeta(k) \leq \zeta(2) = \frac{\pi^2}{6}$.

We look into the structure of \mathcal{M}_k when k is small. We need a theorem of Gundlach [11] and we follow the notations in [2, Theorem 1.39, 1.40]. Note that they considered the group $\mathrm{SL}_2(\mathcal{O})$ instead of $\Gamma = \Gamma_0(\mathfrak{d}, \mathcal{O})$. In particular, $g_k = E_k|_{\alpha_0}$ according to our notations and s_k is a specific cusp form of weight k for $\mathrm{SL}_2(\mathcal{O})$, where

$$\alpha_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{5+\sqrt{5}}{2} \end{pmatrix}.$$

Proposition 7.2. (1) $\mathcal{M}_k = \{0\}$ if k is odd and $\mathcal{M}_k = M_k(\mathrm{SL}_2(\mathcal{O}))|_k \alpha_0^{-1}$ if k is even.
(2) If $k < 20$ is even, then $\mathcal{M}_k = M_k^{\mathrm{sym}}(\mathrm{SL}_2(\mathcal{O}))|_k \alpha_0^{-1}$. In particular, $\bigoplus_{k < 20} \mathcal{M}_k$ is generated by monomials in E_2, E_6 and E_{10} , and we have the following table:

weight k	2	4	6	8	10	12
$\dim(\mathcal{S}_k)$	0	0	1	1	2	3

Proof. We note first that $\mathcal{M}_k = M_k(\Gamma)$ (not $M_k(\mathrm{GL}_2^+(\mathcal{O}))$) and $\mathfrak{d} = \left(\frac{5+\sqrt{5}}{2}\right)$. Therefore,

$$\Gamma = \alpha_0 \mathrm{GL}_2^+(\mathcal{O}) \alpha_0^{-1},$$

and it follows that $\mathcal{M}_k = M_k(\mathrm{GL}_2^+(\mathcal{O}))|_k \alpha_0^{-1}$.

Because $N(\epsilon_0) = -1$, from the definition of \mathcal{M}_k by applying $\epsilon_0 I$, we see that $\mathcal{M}_k = \{0\}$ if k is odd. The same result holds for any $\mathbb{Q}(\sqrt{d})$ with a unit of norm -1 . Note that this is not the case for $\mathrm{SL}_2(\mathcal{O})$.

If k is even, we only have to prove that $M_k(\mathrm{SL}_2(\mathcal{O})) = M_k(\mathrm{GL}_2^+(\mathcal{O}))$. One inclusion is trivial and we assume now $f \in M_k(\mathrm{SL}_2(\mathcal{O}))$. For any $\gamma \in \mathrm{GL}_2^+(\mathcal{O})$, since $\det(\gamma) \gg 0$, we

must have $\det(\gamma) = \epsilon^2$ for some unit ϵ . It follows that

$$f|_k \gamma = f|_k \gamma(\epsilon^{-1}I)(\epsilon I) = f|_k(\epsilon I) = f,$$

because k is even. We are done with (1).

By Gundlach's theorem, in notations of [2, Theorem 1.40], the graded algebra $M_*(\mathrm{SL}_2(\mathcal{O}))$ is generated by g_2, s_5, s_6 and s_{15} . From which we see that if k is even and $k < 20$, then $\mathcal{M}_k = M_k^{\mathrm{sym}}(\mathrm{SL}_2(\mathcal{O}))$. Actually since only s_5 is skew-symmetric among the four generators, the smallest even weight when we can have a nonzero skew-symmetric Hilbert modular form happens at $k = 20$, that is $s_5 s_{15}$. By the structure of $M_{2*}^{\mathrm{sym}}(\mathrm{SL}_2(\mathcal{O}))$ given in [2, Theorem 1.39], the rest of the proposition follows easily. \square

Lemma 7.3. Let h_6 and h_8 be the only cuspidal normalized Hecke eigenforms of weight 6 and 8 respectively, and h_{10}, h'_{10} be the two of weight 10. We have the following Fourier coefficients for these Hecke eigenforms:

\mathfrak{m}	(2)	(3)	\mathfrak{d}	(4)
$c(\mathfrak{m}, h_6)$	20	90	-90	-624
$c(\mathfrak{m}, h_8)$	140	3330	150	3216
$c(\mathfrak{m}, h_{10})$	$170 + 30\sqrt{809}$	$22590 - 540\sqrt{809}$	$570 - 60\sqrt{809}$	$494856 + 10200\sqrt{809}$
$c(\mathfrak{m}, h'_{10})$	$170 - 30\sqrt{809}$	$22590 + 540\sqrt{809}$	$570 + 60\sqrt{809}$	$494856 - 10200\sqrt{809}$

Proof. We first note that

$$\frac{5 + \sqrt{5}}{2} = \mu_1 + \mu_2, \quad \mu_1, \mu_2 \in \mathcal{O}^+$$

has only two solutions

$$\left(\frac{3 + \sqrt{5}}{2}, 1 \right), \quad \left(1, \frac{3 + \sqrt{5}}{2} \right).$$

These decompositions are needed for dealing with the ideal \mathfrak{d} .

Since $E_2 \cdot E_4$ and E_6 have constant terms $(4 \cdot 30 \cdot 4 \cdot 60)^{-1}$ and $67 \cdot (4 \cdot 630)^{-1}$, we must have

$$h_6 = \frac{1}{60} (5360 E_2 \cdot E_4 - 7 E_6).$$

The Fourier coefficients of h_6 can be computed easily from Lemma 4.3. By Proposition 7.2, we have $\dim(\mathcal{S}_8) = 1$ and $h_8 = 120 E_2 \cdot h_6$. The corresponding data follows easily from this.

For the weight 10, it is easy to see that

$$h = \frac{39624096E_2 \cdot E_8 - 3971E_{10}}{30126852}$$

is a normalized cusp form. Clearly $h' = 120E_2 \cdot h_8$ is also a normalized cusp form. We have the following table:

\mathfrak{m}	(2)	(3)	\mathfrak{d}	(4)
$c(\mathfrak{m}, h)$	$\frac{18087260}{119551}$	$\frac{2740912470}{119551}$	$\frac{72616890}{119551}$	$\frac{58400150256}{119551}$
$c(\mathfrak{m}, h')$	260	20970	390	525456

From this and the equation (3.2), we have

$$h_{10} = ah + (1 - a)h', \quad h'_{10} = a'h + (1 - a')h'$$

with $a = 119551(3 - \sqrt{809})/433200$ and a' its conjugate in $\mathbb{Q}(\sqrt{809})$. The normalized Fourier coefficients follow easily from this. \square

Now we are ready to provide and prove the complete list of eigenform product identities when $D = 5$.

Theorem 7.4. The following two identities form the complete list of eigenform product identities $g = f \cdot h$ when $D = 5$ and the weights are 2 or greater (only one of $g = f \cdot h$ and $g = h \cdot f$ is counted):

$$E_4 = 60E_2^2, \quad h_8 = 120E_2 \cdot h_6.$$

Proof. We shall make use of the effective bounds in the proofs of Theorem 4.4 and 4.5.

We first consider products of Eisenstein series. If $k_1 > k_2$, for $|\zeta_F(1 - k_1)/\zeta_F(1 - k_2)| > 1$, we need $k_1 \geq 8$, by (7.1). Using (7.1) and (5.2), we have

$$\begin{aligned} 1 &= \left| (\zeta_F(1 - k_1) + \zeta_F(1 - k_2)) \frac{\zeta_F(1 - k_1 - k_2)}{\zeta_F(1 - k_1)\zeta_F(1 - k_2)} \right| \\ &\geq \left(\frac{6}{\pi^4} \right)^2 \left(\frac{5}{4\pi^2} \right)^{k_2} \frac{\Gamma(k_1 + k_2)^2}{\Gamma(k_1)^2} \left| \left(\frac{6}{\pi^4} \right)^2 \left(\frac{5k_2^2}{4\pi^2} \right)^{k_1 - k_2} - 1 \right|. \end{aligned}$$

Using computer, the right-hand side larger than 1 when $k_1 \geq 8$. So we need to verify the cases $(k_1, k_2) = (4, 2), (6, 2)$ and $(6, 4)$. Note that

$$\zeta_F(-1) = \frac{1}{30}, \quad \zeta_F(-3) = \frac{1}{60}, \quad \zeta_F(-5) = \frac{67}{630}, \quad \zeta_F(-7) = \frac{361}{120}, \quad \zeta_F(-9) = \frac{412751}{1650},$$

and clearly (5.2) does not hold in any of these cases. Now we assume $k_1 = k_2$. Using (7.1) and (5.3), we have

$$1 \geq 1 - 4^{-k_1} = 4^{1-2k_1}(4^{2k_1-1} - 4^{k_1-1}) \geq \frac{\pi}{2} \left(\frac{6}{\pi^4}\right)^3 \left(\frac{5}{4\pi^2}\right)^{\frac{1}{2}} 4^{2-2k_1} \frac{\Gamma(2k_1)^2}{\Gamma(k_1)^2}.$$

The right-hand side is smaller than or equal to 1 only when $k_1 = k_2 = 2$ or 4. The former gives the identity $E_2^2 = \frac{1}{60}E_4$ that holds trivially, while the latter case is impossible since (5.2) does not hold by above zeta values.

Now we consider the case when h is cuspidal. Firstly, since $4\zeta_F(1 - k_1)^{-1}$ is integral by (6.1), $|\zeta_F(1 - k_1)| \leq 4$. By (7.1), we have

$$\frac{72}{\pi^5} \left(\frac{5}{4\pi^2}\right)^{k_1 - \frac{1}{2}} \Gamma(k_1)^2 \leq 4.$$

Such inequality only happens when $k_1 = 2, 4, 6$ or 8. Since $4\zeta_F(1 - k_1)^{-1}$ is integral, from the actual zeta values above, we see that k_1 can only be 2 or 4.

We first assume that $k_1 = 2$. Then $f = E_2$ and by Theorem 4.1,

$$c_1(0, f) = \frac{1}{4}\zeta_F(-1) = \frac{1}{120}, \quad c(\mathfrak{d}, f) = 6, \quad c((2), f) = 5, \quad c((3), f) = 10.$$

Then (6.2) implies

$$15 \cdot 4^{k_2-1} \leq 120^2 + 120 \cdot (2 \cdot 3^{k_2 - \frac{25}{32}} + 5 \cdot 2^{k_2 - \frac{25}{32}} + 4 \cdot 5^{\frac{k_2-1}{2} + \frac{7}{64}} - 22).$$

This holds only if $k_2 \leq 10$. From Proposition 7.2, k_2 can only be 6, 8 or 10. The identity $E_2 \cdot h_6 = \frac{1}{120}h_8$ holds trivially, while $E_2 \cdot h_8$ is not an eigenform by the proof of Lemma 7.3. We need to consider $120E_2 \cdot h_{10}$ and $120E_2 \cdot h'_{10}$. Since we may obtain one from the other by taking conjugate in $\mathbb{Q}(\sqrt{809})$, we just need to consider $h = 120E_2 \cdot h_{10}$. By the table in Lemma 7.3, we easily see that (6.2) does not hold and h is not an eigenform.

Finally, let $k_1 = 4$. We have $f = E_4$ and similarly by Theorem 3.1,

$$c_1(0, f) = \frac{1}{4}\zeta_F(-3) = \frac{1}{240}, \quad c(\mathfrak{d}, f) = 126, \quad c((2), f) = 65, \quad c((3), f) = 730.$$

Then (6.2) implies

$$255 \cdot 4^{k_2-1} \leq 240^2 + 240 \cdot (2 \cdot 3^{k_2 - \frac{25}{32}} + 126 \cdot 2^{k_2 - \frac{25}{32}} + 4 \cdot 5^{\frac{k_2-1}{2} + \frac{7}{64}} - 982),$$

which holds only if $k_2 \leq 8$. Therefore, k_2 can only be 6 or 8. Since $E_4 \cdot h_6$ is a scalar multiple of $E_2 \cdot h_8$, it is not an eigenform from the previous case. Again, for $E_4 \cdot h_8$ from

the table in Lemma 7.3, we check that (6.2) does not hold, forcing $E_4 \cdot h_8$ not to be an eigenform. This completes the proof. \square

8. A GENERAL CONJECTURE

In the light of Theorem 1.7 and of [12] the following conjecture is natural.

Conjecture 8.1. Let $n \geq 1$ be an integer. Then amongst all totally real fields F/\mathbb{Q} with $[F : \mathbb{Q}] = n$ and all nonzero integral ideals \mathfrak{n} , there exist only finitely many solutions to the equation

$$g = f \cdot h,$$

where g, f, h are Hecke eigenforms of level \mathfrak{n} and integral weights 2 or greater.

It also natural to ask if the hypothesis on the degree of the totally real fields considered in conjecture 8.1 is necessary. In other words, perhaps the total number of such identities amongst all totally real fields is finite. But this may be too optimistic at this juncture.

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